Global weak solutions to magnetic fluid flows with nonlinear Maxwell-Cattaneo heat transfer law

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Abstract

We discuss the equations describing the dynamic of the heat transfer in a magnetic fluid flow under the action of an applied magnetic field. Instead of the usual heat transfer equation we use a generalization given by the Maxwell-Cattaneo law which is a system satisfied by the temperature and the heat flux. We prove a global existence of weak solutions to the system having a finite energy.

Keywords : Navier-Stokes equations, Bloch-Torrey equation, magnetostatic equation, Maxwell-Cattaneo law, heat transfer, magnetic field, magnetization
AMS subject classifications: 76N10, 35Q35.

1 Introduction

1.1 Statement of the model

In this work, we study the heat transfer in a magnetic incompressible fluid flow under the action of an applied magnetic field. The temperature θ of the fluid is usually described by the linear heat transfer equation

$$\partial_t \theta + U \cdot \nabla\theta = -\operatorname{div} Q \tag{1}$$

related to the linear Fourier law

$$Q = -\kappa \nabla\theta \tag{2}$$

Q being the heat flux and U the fluid velocity. To ovoid the paradox of the instantaneous heat propagation inherent to the parabolic type equation, another model was offered in the pioneering work of Vernotte [23] and Cattaneo [6]. In this model, the Fourier law (2) is replaced by the heat-flux equation

$$\tau \partial_t Q + Q = -\kappa \nabla\theta \tag{3}$$

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where $\tau > 0$ is the time relaxation parameter. For $\tau = 0$, we recover equation (2). Combining the temperature equation and the heat-flux equation we see that θ satisfies an hyperbolic type equation. System (1)-(3) was generalized by Guyer and Krumhansl see [13] for example, by introducing a diffusion process in (3) so that the heat-flux equation becomes

$$\tau(\partial_t Q - \gamma \Delta Q) = -Q - \kappa \nabla \theta \tag{4}$$

where $\gamma > 0$ is a diffusion coefficient. When the heat conductivity is enhanced by radiation effects see [12, 10, 11], the linear Fourier law is replaced by a nonlinear one which writes aqs

$$Q = -\nabla \mathcal{K}(\theta). \tag{5}$$

In [11], the model of heat transfer by the nonlinear Fourier law in an incompressible fluid flow has been discussed.

In this work we are dealing with the nonlinear Maxwell-Cattaneo law for heat transfer which is a generalization of the nonlinear Fourier law, more precisely we consider that the dynamic of the couple (θ, Q) is governed by the system

$$\begin{aligned} &\partial_t \theta + U \cdot \nabla \theta = -\operatorname{div} Q \\ &\tau(\partial_t Q - \gamma \Delta Q) = -\frac{\tau}{2} \operatorname{curl} U \times Q - Q - \nabla \mathcal{K}(\theta). \end{aligned} \tag{6}$$

The monotone function $\mathcal{K}(\theta)$ discussed in this work is given by

$$\mathcal{K}(\theta) = \kappa\, \theta + \alpha\, \theta^3 \tag{7}$$

where $\kappa > 0$ and $\alpha > 0$ are the heat conductivity coefficients and we refer the reader to [19, 20] for the introduction of the rotation term $\frac{1}{2}\operatorname{curl} U \times Q$. Notice that the power 3 used in the definition of the function $\mathcal{K}$ is less than the values indicated in [12].

The Maxwell-Cattaneo system (6) is coupled to the incompressible Navier-Stokes equations satisfied by the fluid velocity U and the pressure p as well as to the Bloch-Torrey equation satisfied by the magnetization field M and the magnetostatic equation for the magnetic field H. Namely, we have

$$\begin{aligned} &\operatorname{div} U = 0 \\ &\partial_t U + (U.\nabla)U - \eta \Delta U + \nabla p = -\rho(\theta) g + \mu_0 (M.\nabla) H + \tfrac{\mu_0}{2} \operatorname{curl}(M \times H) \\ &\partial_t M + (U.\nabla) M - \sigma \Delta M + \frac{1}{\delta}(M - \chi_0 H) = \frac{1}{2} \operatorname{curl} U \times H - \beta_0\, M \times (M \times H) \\ &\operatorname{div}(H + M) = F, \quad \operatorname{curl} H = 0 \end{aligned} \tag{8}$$

where the density $\rho(\theta)$ is given by the state of law

$$\rho(\theta) = \rho_0 (1 - \beta(\theta - \theta^0)) \tag{9}$$

where ρ_0 is the fluid density at the the temperature θ^0 and β is a physical coefficient. The function g represents the force of gravity, F is a function linked to the applied magnetic field and $\eta, \mu_0, \sigma, \delta, \chi_0, \beta_0 > 0$ are physical parameters.

When the magnetization M is assumed to be in equilibrium state meaning it is parallel to the magnetic field H, the model in consideration is quite different from the one studied in this work. The magnetization law writes in general as

$$M = \chi(\theta, |H|)H \tag{10}$$

In that case, the Maxwell-Cattaneo system becomes

$$\begin{aligned} \partial_t\theta + U\cdot\nabla\theta + \mu_0\theta\,\frac{\partial M}{\partial\theta}\cdot(U\cdot\nabla)H &= -\mathrm{div}\,Q + \eta\Phi(U) \\ \tau\Big(\partial_t Q - \Delta Q + \tfrac{1}{2}\mathrm{curl}\,U\times Q\Big) &= -Q - \nabla\mathcal{K}(\theta) \end{aligned} \tag{11}$$

where $\mu_0\,\theta\,\dfrac{\partial M}{\partial\theta}\cdot(U\cdot\nabla)H$ is the thermal power and $\Phi(U)$ is the energy dissipation. The heat transfer problem in an incompressible fluid flow under the above Maxwell-Cattaneo law in a magnetic fluid is an open problem.

Let $D\subset\mathbb{R}^3$ be an open, bounded, regular and simply connected domain, with boundary Γ. For $T>0$ fixed, we set $D_T = (0,T)\times D$ and $\Gamma_T = (0,T)\times\Gamma$. The equations (6) and (8) are set on D_T with the following initial and boundary conditions

$$\begin{aligned} &U(0) = U_0,\ \mathrm{div}\,U_0 = 0,\ M(0) = M_0,\ \text{in } D \\ &U = 0,\ M\cdot\mathbf{n} = 0,\ \mathrm{curl}\,M\times\mathbf{n} = 0,\ H\cdot\mathbf{n} = 0, \text{on } \Gamma_T \end{aligned} \tag{12}$$

$$\begin{aligned} &\theta(0) = \theta_0,\ Q(0) = Q_0 \text{ in } D \\ &Q\times\mathbf{n} = 0,\ \tau\gamma\mathrm{div}\,Q - \mathcal{K}(\theta) = 0 \text{ on } \Gamma_T \end{aligned} \tag{13}$$

where $\mathbf{n}$ represents the unit outward normal to the boundary Γ. Problem (6)-(8)-(12)-(13) will be labeled problem $(\mathcal{P})$.

System (8) with the temperature equation (1) has been discussed in [1, 2]. The linear Maxwell-Cattaneo system (1)-(3) has been studied in [14, 15] in the case where the velocity U is fixed.

1.2 Notations and spaces

For $1\le q\le\infty$ and $s\in\mathbb{R}$, let $L^q(D)$ and $W^{s,q}(D)$ be the usual Lebesgue and Sobolev spaces of scalar functions. If $q=2$, $W^{s,2}(D)$ is denoted by $H^s(D)$ and $\|\cdot\|$ and $(\cdot;\cdot)$ denote the norm and the scalar product of the Hilbert space $L^2(D)$. For vector valued functions we use the notations $\mathbb{L}^q(D)$, $\mathbb{W}^{s,q}(D)$, $\mathbb{H}^s(D)$ and the notations of norm and the scalar product of $\mathbb{L}^2(D)$ are unchanged. If $\mathcal{V}$ is a Banach space we denote by $\langle\cdot;\cdot\rangle_{\mathcal{V}'\times\mathcal{V}}$ (or simply $\langle\cdot;\cdot\rangle$ if no confusion arises) the duality product where $\mathcal{V}'$ is the dual space of $\mathcal{V}$. If V is an Hilbert space with scalar product $(\cdot;\cdot)$, we set

$$\mathcal{C}([0,T];V \text{ weak}) = \{u : [0,T]\to V; (u(\cdot),v)\in\mathcal{C}([0,T]),\ \forall v\in V\}.$$

Let $\mathcal{D}(D,\mathbb{R}^3)$ the set of functions $f : D \to \mathbb{R}^3$ which are infinitely differentiable with compact support in D and $\mathbb{H}^1_0(D)$ its closure in $\mathbb{H}^1(D)$. Now, we introduce the functional spaces used in the theory of Navier-Stokes equations, see [22, 7] for example

$$\mathcal{D}_s(D) = \{v \in \mathcal{D}(D,\mathbb{R}^3);\ \operatorname{div} v = 0 \text{ in } D\}$$

$$\mathcal{U} = \text{closure of } \mathcal{D}_s(D) \text{ in } \mathbb{H}^1(D),\ \mathcal{U}_0 = \text{closure of } \mathcal{D}_s(D) \text{ in } \mathbb{L}^2(D).$$

Then it is well known that

$$\begin{aligned} \mathcal{U} &= \{v \in \mathbb{H}^1_0(D);\ \operatorname{div} v = 0 \text{ in } D\} \\ \mathcal{U}_0 &= \{v \in \mathbb{L}^2(D);\ \operatorname{div} v = 0 \text{ in } D,\ v\cdot\mathbf{n} = 0 \text{ on } \Gamma\} \end{aligned} \tag{14}$$

and identifying $\mathcal{U}_0$ with its dual, we get as usual the inclusions $\mathcal{U} \subset \mathcal{U}_0 \subset \mathcal{U}'$.
For the Bloch-Torrey equation satisfied by M and the heat-flux equation satisfied by Q we introduce the Hilbert spaces

$$\mathbb{H}^1_t(D) = \{M \in \mathbb{H}^1(D);\ M\cdot\mathbf{n} = 0 \text{ on } \Gamma\}$$

$$\mathbb{H}^1_n(D) = \{Q \in \mathbb{H}^1(D);\ Q\times\mathbf{n} = 0 \text{ on } \Gamma\}$$

equipped with the norm of $\mathbb{H}^1(D)$. Then (see [7] for example) there exists $C > 0$ such that for all V in either $\mathbb{H}^1_t(D)$ or $\mathbb{H}^1_n(D)$ the following estimate holds

$$\|\nabla V\| \le C(\|V\|^2 + \|\operatorname{curl} V\|^2 + \|\operatorname{div} V\|^2)^{1/2} \tag{15}$$

hence the norm of $\mathbb{H}^1(D)$ is equivalent to the norm $(\|V\|^2 + \|\operatorname{curl} V\|^2 + \|\operatorname{div} V\|^2)^{1/2}$ on the spaces $\mathbb{H}^1_t(D)$ and $\mathbb{H}^1_n(D)$. We recall the relation $-\Delta = \operatorname{curl}^2 - \nabla\operatorname{div}$ so that for regular vector fields Ψ and Φ the following Green formula holds

$$\begin{aligned} -\int_D \Delta\Psi\cdot\Phi\,dx &= \int_D \operatorname{curl}\Psi\cdot\operatorname{curl}\Phi\,dx + \int_D \operatorname{div}\Psi\operatorname{div}\Phi\,dx \\ &\quad + \int_\Gamma \operatorname{curl}\Psi\cdot(\Phi\times\mathbf{n})\,d\Gamma - \int_\Gamma \operatorname{div}\Psi(\Phi\cdot\mathbf{n})\,d\Gamma. \end{aligned}$$

To deal with the magnetostatic equation, we set

$$L^2_\sharp = \{\psi \in L^2(D);\ \int_D \psi(x)\ dx = 0\} \text{ and } H^1_\sharp = H^1(D) \cap L^2_\sharp.$$

The Hilbert space $H^1_\sharp$ is equipped with the norm $\|\nabla\psi\|$ which is equivalent to the usual norm of $H^1(D)$ thanks to Poincaré-Wirtinger inequality : there exists $C > 0$ such that for all $\psi \in \mathbb{H}^1_\sharp$ we have

$$\|\psi\| \le C\|\nabla\psi\|. \tag{16}$$

To end these notations, we point out that throughout this paper, $C > 0$ indicates a generic constant depending only on some bounds of the physical data, which takes different values in different occurrences. The dependency of the constants $C > 0$ with respect to a parameter m is written as C_m.

Now, let us focus our attention on the magnetostatic equation to give some useful continuity results on the solution

1.3 The magnetostatic equation

Let $M \in \mathbb{L}^2(D)$ and $F \in L^2_\sharp$, we consider the following problem

$$\text{Find } \varphi \in H^1_\sharp;\ \ \forall \psi \in H^1_\sharp,\ \ \int_D (\nabla\varphi + M)\cdot\nabla\psi\, dx = -\int_D F\psi\, dx. \tag{17}$$

This problem admits a unique solution φ in $H^1_\sharp$ and we have

$$\int_D \nabla\varphi.M\ dx = -\|\nabla\varphi\|^2 - \int_D F\varphi\ dx. \tag{18}$$

then

$$\|\nabla\varphi\| \le (\|M\| + C\|F\|). \tag{19}$$

In particular the application

$$\mathcal{H} : (M, F) \mapsto \varphi \tag{20}$$

is continuous from $\mathbb{L}^2(D) \times L^2_\sharp$ to $H^1_\sharp$. Furthermore testing equation (17) with $\psi - \int_D \psi\ dx, \psi \in H^1(D)$, we see that

$$\int_D (\nabla\varphi + M)\cdot\nabla\psi\, dx = -\int_D F\psi\, dx, \quad \forall\psi \in H^1(D) \tag{21}$$

and $H = \nabla\varphi$ solves the problem

$$\begin{cases} \operatorname{div}(H + M) = F, \quad \operatorname{curl} H = 0 \text{ in } D \\ (H + M)\cdot\mathbf{n} = 0 \text{ on } \Gamma. \end{cases}$$

Moreover using classical regularity results for elliptic problems, we conclude that if $F \in L^2_\sharp$ and $M \in \mathbb{H}^1_t(D)$, then $\varphi \in \mathbb{H}^2(D) \cap H^1_\sharp$ and we have

$$\|\varphi\|_{H^2(D)} \le C(\|\operatorname{div} M\| + \|F\|). \tag{22}$$

Therefore $H = \nabla\varphi \in \mathbb{H}^1_t(D)$ and we have

$$\|H\|_{H^1(D)} \le C(\|\operatorname{div} M\| + \|F\|). \tag{23}$$

We can see that $\mathcal{H}$ is also continuous from $\mathbb{L}^2(D_T) \times L^2(0,T;L^2_\sharp)$ to $L^2(0,T;H^1_\sharp)$ and from $H^1(0,T;\mathbb{L}^2(D)) \times H^1(0,T;L^2_\sharp)$ to $H^1(0,T;H^1_\sharp)$. Moreover for $F \in H^1(0,T;L^2_\sharp)$ and $M \in H^1(0,T;\mathbb{L}^2(D))$, we have

$$\int_D (\nabla(\partial_t\varphi) + \partial_t M)\cdot\nabla\psi\, dx = -\int_D \partial_t F\psi\, dx, \forall\psi \in H^1(D), \quad t \in (0,T). \tag{24}$$

2 Main results

Before stating our main result, let us give the formal energy estimates for problem $(\mathcal{P})$.

2.1 Energy estimates

Let (U, M, H, θ, Q) be a regular solution to system $(\mathcal{P})$. We proceed as in [1, 2] to obtain, for θ fixed, the energy estimate satisfied by (U, M, H). For $t \in [0, T]$, we set

$$\begin{aligned}\mathcal{E}_{nsbt}(t) &= \frac{1}{2}\|U(t)\|^2 + \frac{\mu_0}{2}(\|M(t)\|^2 + \|H(t)\|^2) \\ \mathcal{E}_{nsbt,0} &= \frac{1}{2}\|U_0\|^2 + \frac{\mu_0}{2}(\|M_0\|^2 + \|H_0\|^2)\end{aligned} \tag{25}$$

where $H_0 = \nabla\varphi_0$ and φ_0 is the unique solution of the following problem (see subsection 1.3)

$$\text{Find } \varphi_0 \in H^1_\sharp \text{ such that}$$

$$\int_D (\nabla\varphi_0 + M_0)\cdot\nabla\psi\, dx = -\int_D F(0)\psi\, dx, \quad \forall\psi \in H^1_\sharp \tag{26}$$

and

$$\begin{aligned}\mathcal{F}_{nsbt}(t) = \eta\|\nabla U(t)\|^2 + \mu_0\sigma(\|\mathrm{curl}\, M(t)\|^2 + 2\|\mathrm{div}\, M(t)\|^2) + \frac{\mu_0}{\delta}\|M(t)\|^2 + \\ \frac{\mu_0}{\delta}(1 + 2\chi_0)\|H(t)\|^2 + \beta_0\mu_0\|M(t)\times H(t)\|^2.\end{aligned} \tag{27}$$

Then we get the energy estimate

$$\mathcal{E}_{nsbt}(t) + \int_0^t \mathcal{F}_{nsbt}(s)\, ds \le \mathcal{E}_{nsbt,0} + C\int_0^t \|\rho(\theta(s))\|^2\, ds + C\int_0^t \|G(s)\|^2\, ds \tag{28}$$

for all $t \ge 0$ where

$$G(t) = \|F(t)\|^2 + \|\partial_t F(t)\|^2. \tag{29}$$

Now we consider the Maxwell-Cattaneo system (6) satisfied by (θ, Q) for U fixed. Let ϖ the primitive function of $\mathcal{K}$ defined by

$$\varpi(\theta) = \frac{\kappa}{2}\theta^2 + \frac{\alpha}{4}\theta^4. \tag{30}$$

Multiplying the temperature equation by $\mathcal{K}(\theta)$ and the heat-flux equation by Q then integrating by parts and adding both results, we get the energy estimate associated with the Maxwell-Cattaneo system

$$\mathcal{E}_{mc}(t) + \int_0^t \mathcal{F}_{mc}(s)\, ds \le \mathcal{E}_{mc,0} \tag{31}$$

for all $t \ge 0$ with

$$\mathcal{E}_{mc}(t) = \int_D \varpi(\theta(t))\, dx + \frac{\tau}{2}\|Q(t)\|^2, \quad \mathcal{E}_{mc,0} = \int_D \varpi(\theta_0)\, dx + \frac{\tau}{2}\|Q_0\|^2 \tag{32}$$

$$\mathcal{F}_{mc}(t) = \tau\gamma(\|\mathrm{curl}\, Q(t)\|^2 + \|\mathrm{div}\, Q(t)\|^2) + \|Q(t)\|^2. \tag{33}$$

The total energy $\mathcal{E}$ and the total dissipation energy $\mathcal{F}$ of the full problem $(\mathcal{P})$ are defined by

$$\mathcal{E}(t) = \mathcal{E}_{nsbt}(t) + \mathcal{E}_{mc}(t), \quad \mathcal{F}(t) = \mathcal{F}_{nsbt}(t) + \mathcal{F}_{mc}(t) \tag{34}$$

and it holds

$$\mathcal{E}(t) + \int_0^t \mathcal{F}(s)\, ds \le \mathcal{E}_0 + C\int_0^t \|\rho(\theta(s))\|^2\, ds + C\int_0^t G(s)\, ds. \tag{35}$$

2.2 Statement of the result

We will use the following hypotheses

$$
\begin{aligned}
U_0 \in \mathcal{U}_0,\ \ M_0,\, Q_0 \in \mathbb{L}^2(D),\ \operatorname{div} Q_0 \in L^{12/11}(D),\ \theta_0 \in L^4(D) \\
g \in \mathbb{L}^\infty(D_T),\quad F \in H^1(0,T;L^2(D)),\ \int_D F(t,x)\,dx = 0 \text{ for all } t \in [0,T].
\end{aligned} \tag{36}
$$

Let us give now the definition of a global weak solution to problem $(\mathcal{P})$

Definition 1 *We say that (U, M, H, θ, Q) is a global weak solution with finite energy of problem $(\mathcal{P})$ if the following conditions are fulfilled*

$$
\begin{aligned}
& U \in L^\infty(0,T;\mathcal{U}_0) \cap L^2(0,T;\mathcal{U}) \\
& M \in L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1_t(D)) \\
& H \in L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1_t(D)) \\
& Q \in L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1_n(D)) \\
& \theta \in L^\infty(0,T;L^4(D))
\end{aligned} \tag{37}
$$

and

(i) the linear momentum equation holds weakly in the sense that for all $v \in \mathcal{U}$

$$
\begin{gathered}
\frac{d}{dt}\int_D U\cdot v\,dx + \int_D (U\cdot\nabla)U\cdot v\,dx + \eta\int_D \nabla U\cdot\nabla v\,dx = \\
-\int_D \rho(\theta) g\cdot v\,dx + \mu_0\int_D (M\cdot\nabla)H\cdot v\,dx + \frac{\mu_0}{2}\int_D M\times H\cdot \operatorname{curl} v\,dx \\
U(0) = U_0
\end{gathered} \tag{38}
$$

(ii) the magnetization equation satisfies for all $w \in \mathbb{H}^1_t(D)$ the weak formulation

$$
\begin{gathered}
\frac{d}{dt}\int_D M\cdot w\,dx + \int_D (U\cdot\nabla)M\cdot w\,dx + \sigma\int_D \operatorname{curl} U\cdot\operatorname{curl} w\,dx \\
+\sigma\int_D \operatorname{div} M\operatorname{div} w\,dx + \frac{1}{\delta}\int_D (M - \chi_0 H)\cdot w\,dx = \\
\frac{1}{2}\int_D \operatorname{curl} U\times H\cdot w\,dx - \beta_0\int_D M\times H\cdot M\times w\,dx \\
M(0) = M_0
\end{gathered} \tag{39}
$$

(iii) the magnetic field is given by $H = \nabla\varphi$ where $\varphi \in L^\infty(0,T;L^2_\sharp)$ and satisfies for all $\psi \in H^1_\sharp$

$$
\int_D (\nabla\varphi(t) + M(t))\cdot\nabla\psi\,dx = -\int_D F(t)\psi\,dx \tag{40}
$$

(iv) the couple (θ, Q) satisfies the Maxwell-Cattaneo system in the following sense

$$
\int_{D_T} \theta(\partial_t a + U\cdot\nabla a)\,dxdt = \int_{D_T} \operatorname{div} Q\, a\,dxdt - \int_D \theta_0\, a(0)\,dx \tag{41}
$$

for all $a \in \mathcal{D}([0,T[\times\overline{D})$ *and for all* $b \in \mathbb{H}^1_n(D)$ *with* $\operatorname{div} b \in L^4(D)$

$$\begin{gathered}\tau \frac{d}{dt}\int_{D_T} Q\cdot b\,dx + \tau\gamma\int_D (\operatorname{curl} Q\cdot\operatorname{curl} b + \operatorname{div} Q\operatorname{div} b)\,dx +\\ \int_D Q\cdot b\,dx + \frac{\tau}{2}\int_D \operatorname{curl} U\times Q\cdot b\,dx = \int_D \mathcal{K}(\theta)\operatorname{div} b\,dx \\ Q(0) = Q_0.\end{gathered} \tag{42}$$

Moreover the energy estimates (28) and (31) hold for all $t \in (0,T)$*.*

Remark 1

1. *As usual, we get the pressure* $p \in W^{-1,\infty}(0,T;L^2(D))$ *by using the De Rham theorem.*
2. *From the weak formulations, we deduce that* $(\partial_t U, \partial_t M, \partial_t Q) \in L^1(0,T;\mathcal{U}'\times (H^1_t(D))'\times(H^1_n(D))')$ *so that* $(U,M,Q) \in \mathcal{C}([0,T];\mathcal{U}'\times(H^1_t(D))'\times(H^1_n(D))')$ *and the corresponding initial conditions are meaningful and moreover* $U, M, Q \in \mathcal{C}([0,T];\mathbb{L}^2(D)$ *weak).*
3. *The theory of transport equation leads to the result* $\theta \in \mathcal{C}([0,T];\mathbb{L}^4(D)$ *weak)* $\cap$ $\mathcal{C}([0,T];\mathbb{L}^p(D))$, *for all* $1 \le p < 4$ *(see [5] for example) which gives a sense to the initial condition.*

Theorem 1 *Under hypotheses (36), there exists a global weak solution with finite energy of problem* $(\mathcal{P})$*. Moreover* θ *has the regularity*

$$\theta \in L^{36/11}(0,T;L^{36/7}(D)). \tag{43}$$

Remark 2 *One can relax the condition* $\operatorname{div} b \in L^4(D)$ *on test functions* b *in (42) to the condition* $\operatorname{div} b \in L^{12/5}(D)$*.*

We will prove existence of solutions to problem $(\mathcal{P})$ in several steps, using a regularization method and some compactness results. The paper is organized as follows.
In section 3, we introduce the regularized problem $(\mathcal{P}_\nu)$ obtained by adding an elliptic term $-\nu\nabla\cdot(|\nabla\theta|^2\nabla\theta)$ in the temperature equation, $\nu > 0$ being a small parameter together to a regularization of the initial condition θ_0. By using the Faedo-Galerkine method, we obtain a sequence of approximated solutions $(U_n, M_n, H_n, \theta_n, Q_n)$ which converge towards $(U_\nu, M_\nu, H_\nu, \theta_\nu, Q_\nu)$ a global weak solution with finite energy of system $(\mathcal{P}_\nu)$.
In section 4, we prove Theorem 1. We first introduce an auxiliary problem satisfied by $\zeta_\nu = \tau\gamma\operatorname{div} Q_\nu - \mathcal{K}(\theta_\nu)$ and establish a compacity result verified by ζ_ν which allows to get the limit of the nonlinear term $\mathcal{K}(\theta_\nu)$. Then we get Theorem 1 by passing to the limit as $\nu \to 0$.

3 The regularized problem $(\mathcal{P}_\nu)$

Let $\nu > 0$ be a small parameter and (θ_0^ν) such that

$$(\theta_0^\nu) \subset W^{1,4}(D), \quad \theta_0^\nu \to \theta_0 \text{ strongly in } L^4(D). \tag{44}$$

We define the regularized problem $(\mathcal{P}_\nu)$ as the system $(8)-(12)$ coupled to the regularized Maxwell-Cattaneo system

$$\begin{aligned}
&\partial_t\theta + (U\cdot\nabla)\theta - \nu\nabla\cdot(|\nabla\theta|^2\nabla\theta) = -\operatorname{div} Q \text{ in } D_T \\
&\tau(\partial_t Q - \gamma\Delta Q) = -\frac{\tau}{2}\operatorname{curl} U \times Q - Q - \nabla\mathcal{K}(\theta) \text{ in } D_T \\
&\nu|\nabla\theta|^2\nabla\theta\cdot\mathbf{n} = 0,\ Q\times\mathbf{n} = 0,\ \tau\gamma\operatorname{div} Q - \mathcal{K}(\theta) = 0 \text{ on } \Gamma_T \\
&\theta(0) = \theta_0^\nu,\ Q(0) = Q_0 \text{ in } D
\end{aligned} \tag{45}$$

Note that we use the nonlinear elliptic operator $-\nu\nabla\cdot(|\nabla\theta|^2\nabla\theta)$ instead of $-\nu\Delta\theta$ which is commonly used to regularize a transport equation, owing to obtain approximate solutions θ_ν belonging to $W^{1,4}(D)$ and therefore to $L^\infty(D)$.
Proceeding as previously the energy associated with (45) takes the form

$$\mathcal{E}_{mc}(t) + \int_0^t \mathcal{F}_{mc}(s)\,ds + \nu\int_0^t \mathcal{R}(s)\,ds \le \mathcal{E}_{mc,0}^\nu \tag{46}$$

for all $t \ge 0$ where

$$\mathcal{R}(t) = \kappa\|\nabla\theta\|_{L^4(D)}^4 + 3\alpha\int_D \theta^2|\nabla\theta|^4\,dx \tag{47}$$

which is well defined thanks to the Sobolev embedding $W^{1,4}(D) \subset \mathcal{C}(\overline{D})$ and

$$\mathcal{E}_{mc,0}^\nu = \int_D (\frac{\kappa}{2}|\theta_0^\nu|^2 + \int_D \frac{\alpha}{4}|\theta_0^\nu|^4)\,dx + \|Q_0\|^2$$

It is easy to verify that the energy estimate associated with the problem $(\mathcal{P}_\nu)$ writes as

$$\mathcal{E}(t) + \int_0^t \mathcal{F}(s)\,ds + \nu\int_0^t \mathcal{R}(s)\,ds \le C + C\int_0^t \|\rho(\theta(s))\|^2\,ds + C\int_0^t \|G(s)\|^2\,ds \tag{48}$$

where $C > 0$ does not depend on ν. We will prove the following existence result

Theorem 2 *Under hypotheses (36), there exists a global weak solution* $(U_\nu, M_\nu, H_\nu, \theta_\nu, Q_\nu)$ *of problem* $(\mathcal{P}_\nu)$ *such that*

$$\begin{aligned}
&U_\nu \in L^\infty(0,T;\mathcal{U}_0) \cap L^2(0,T;\mathcal{U}) \\
&M_\nu,\ H_\nu \in L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}_t^1(D)) \\
&Q_\nu \in L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}_n^1(D)) \\
&\theta_\nu \in L^\infty(0,T;L^4(D)) \cap L^4(0,T;W^{1,4}(D))
\end{aligned} \tag{49}$$

and satisfying the energy estimates (46) and (48) and the problem in the following sense *(i)* $U_\nu(0) = U_0$ *and for all* $v \in \mathcal{U}$

$$\frac{d}{dt}\int_D U_\nu \cdot v\,dx + \int_D (U_\nu\cdot\nabla)U_\nu\cdot v\,dx + \eta\int_D \nabla U_\nu\cdot\nabla v\,dx = \\ -\int_D \rho(\theta_\nu)g\cdot v\,dx + \mu_0\int_D (M_\nu\cdot\nabla)H_\nu\cdot v\,dx + \frac{\mu_0}{2}\int_D M_\nu\times H_\nu\cdot\operatorname{curl} v\,dx \tag{50}$$

(ii) $M_\nu(0) = M_0$ *and for all* $w \in \mathbb{H}^1_t(D)$

$$\begin{aligned}\frac{d}{dt}\int_D M_\nu\cdot w\,dx + \int_D (U_\nu\cdot\nabla)M_\nu\cdot w\,dx + \sigma\int_D \operatorname{curl} M_\nu\cdot\operatorname{curl} w\,dx \\ +\sigma\int_D \operatorname{div} M_\nu \operatorname{div} w\,dx + \frac{1}{\delta}\int_D (M_\nu - \chi_0 H_\nu)\cdot w\,dx = \\ \frac{1}{2}\int_D \operatorname{curl} U_\nu\times H_\nu\cdot w\,dx - \beta_0\int_D M_\nu\times H_\nu\cdot M_\nu\times w\,dx\end{aligned} \tag{51}$$

(iii) $\theta_\nu(0) = \theta_0^\nu$ *and for all* $a \in W^{1,4}(D)$

$$\frac{d}{dt}\int_D \theta_\nu a\,dx - \int_D \theta_\nu U\cdot\nabla a\,dx + \nu\int_D |\nabla\theta_\nu|^2\nabla\theta_\nu\cdot\nabla a\,dx = -\int_D \operatorname{div} Q_\nu\, a\,dx \tag{52}$$

(iv) $Q_\nu(0) = Q_0$ *and for all* $b \in \mathbb{H}^1_n(D)$

$$\begin{aligned}\tau\frac{d}{dt}\int_D Q_\nu\cdot b\,dx + \tau\gamma\int_D (\operatorname{curl} Q_\nu\cdot\operatorname{curl} b + \operatorname{div} Q_\nu \operatorname{div} b)\,dx + \\ \frac{\tau}{2}\int_D \operatorname{curl} U_\nu\times Q_\nu\cdot b\,dx = -\int_D Q_\nu\cdot b\,dx + \int_D \mathcal{K}(\theta_\nu)\operatorname{div} b\,dx\end{aligned} \tag{53}$$

with $H_\nu = \nabla\varphi_\nu$ *where* $\varphi_\nu = \mathcal{H}(M_\nu, F)$ *is defined in (20).*

3.1 Faedo-Galerkine approximation for $(\mathcal{P}_\nu)$

Let $\nu > 0$ be fixed, consider the weak formulation of problem $(\mathcal{P}_\nu)$ given in Theorem 2. In order to solve this problem by the Faedo-Galerkine method, we introduce the Hilbert basis $(V_j)_{j\geq 1}$, $(W_j)_{j\geq 1}$, $(\Phi_j)_{j\geq 1}$ of the spaces $\mathcal{U}$, $\mathbb{H}^1_t(D)$, $\mathbb{H}^1_n(D)$ respectively and a basis $(v_j)_{j\geq 1}$ of $W^{1,4}(D)$. For simplicity, we assume these basis to be orthonormal in $L^2(D)$. We seek for approximated solutions of the system $(\mathcal{P}_\nu)$ of the form

$$\begin{aligned}U_n(t) = \sum_{j=1}^n \alpha_j(t)V_j, \quad M_n(t) = \sum_{j=1}^n \beta_j(t)W_j, \\ \theta_n(t) = \sum_{j=1}^n a_j(t)v_j, \quad Q_n(t) = \sum_{j=1}^n b_j(t)\Phi_j\end{aligned} \tag{54}$$

satisfying for all $n \in \mathbb{N}^*$ and $1 \leq j \leq n$

$$\begin{aligned}\text{(i)}\quad \frac{d}{dt}\int_D U_n\cdot V_j\,dx + \int_D (U_n\cdot\nabla)U_n\cdot V_j\,dx + \eta\int_D \nabla U_n\cdot\nabla V_j\,dx = \\ -\int_D \rho(\theta_n)g\cdot V_j\,dx + \mu_0\int_D (M_n\cdot\nabla)H_n\cdot V_j\,dx + \frac{\mu_0}{2}\int_D M_n\times H_n\cdot\operatorname{curl} V_j\,dx \\ U_n(0) = U_{0n}\end{aligned} \tag{55}$$

$$\text{(ii)} \quad \frac{d}{dt}\int_D M_n\cdot W_j\,dx+\int_D (U_n\cdot\nabla)M_n\cdot W_j\,dx+\sigma\int_D \operatorname{curl} M_n\cdot\operatorname{curl} W_j\,dx$$
$$+\sigma\int_D \operatorname{div} M_n\operatorname{div} W_j\,dx+\frac{1}{\delta}\int_D (M_n-\chi_0 H_n)\cdot W_j\,dx= \tag{56}$$
$$\frac{1}{2}\int_D \operatorname{curl} U_n\times H_n\cdot W_j\,dx-\beta_0\int_D M_n\times H_n\cdot M_n\times W_j\,dx$$
$$M_n(0)=M_{0n}$$

$$\text{(iii)} \quad \frac{d}{dt}\int_D \theta_n v_j\,dx-\int_D \theta_n U_n\cdot\nabla v_j\,dx+\nu\int_D |\nabla\theta_n|^2\nabla\theta_n\cdot\nabla v_j\,dx=$$
$$-\int_D \operatorname{div} Q_n\, v_j\,dx \tag{57}$$
$$\theta_n(0)=\theta_{0n}^{\nu}$$

$$\text{(iv)} \quad \tau\frac{d}{dt}\int_D Q_n\cdot\Phi_j\,dx+\tau\gamma\int_D (\operatorname{curl} Q_n\cdot\operatorname{curl}\Phi_j+\operatorname{div} Q_n\operatorname{div}\Phi_j)\,dx=$$
$$-\frac{\tau}{2}\int_D \operatorname{curl} U_n\times Q_n\cdot\Phi_j\,dx-\int_D Q_n\cdot\Phi_j\,dx+\int_D \mathcal{K}(\theta_n)\operatorname{div}\Phi_j\,dx \tag{58}$$
$$Q_n(0)=Q_{0n}$$

where

$$H_n=\nabla\varphi_n,\quad \varphi_n=\mathcal{H}(M_n,F))$$

$$U_{0n}=\sum_{j=1}^n \alpha_{0n}^j V_j,\quad M_{0n}=\sum_{j=1}^n \beta_{0n}^j W_j,$$
$$\theta_{0n}^{\nu}=\sum_{j=1}^n a_{0n}^{\nu,j}\ v_j,\quad Q_{0n}=\sum_{j=1}^n b_{0n}^j\Phi_j.$$

We assume that

$$(U_{0n},M_{0n},Q_{0n})\to(U_0,M_0,Q_0)\text{ strongly in }(\mathbb{L}^2(D))^3$$
$$\theta_{0n}^{\nu}\to\theta_0^{\nu}\text{ strongly in }W^{1,4}(D). \tag{59}$$

This problem will be labeled $(\mathcal{P}_\nu^n)$.

3.2 Solving the system $(\mathcal{P}_\nu^n)$

Let the vector valued functions $\alpha^n=(\alpha_1,\cdots,\alpha_n)$, $\beta^n=(\beta_1,\cdots,\beta_n)$, $a^n=(a_1,\cdots,a_n)$ and $b^n=(b_1,\cdots,b_n)$, we consider the function

$$t\in[0,T]\to\gamma_n(t)=(\alpha^n(t),\beta^n(t),a^n(t),b^n(t))\in(\mathbb{R}^n)^4$$

then γ_n satisfies the ordinary differential system

$$\gamma_n'+A_n\gamma_n=Z_n(t,\gamma_n),\quad \gamma_n(0)=\gamma_{0n} \tag{60}$$

where $\gamma_{0n}=(\alpha_{0n},\beta_{0n},a_{0n}^{\nu},b_{0n})\in(\mathbb{R}^n)^4$, A_n is a $n^4\times n^4$ constant matrix involving the terms

$$\eta\int_D \nabla V_i\cdot\nabla V_j\ dx,\quad \sigma\int_D (\operatorname{curl} W_i\cdot\ \operatorname{curl} W_j+\operatorname{div} W_i\operatorname{div} W_j)\,dx$$
$$\tau\gamma\int_D (\operatorname{curl}\Phi_i\cdot\operatorname{curl}\Phi_j+\operatorname{div}\Phi_i\operatorname{div}\Phi_j)\,dx+\int_D \Phi_i\cdot\Phi_j\,dx$$

and the vector field $Z_n = (Z_n^1, Z_n^2, Z_n^3, Z_n^4) \in (\mathbb{R}^n)^4$ is defined as follows

$$\begin{aligned}
Z_{nj}^1(t,\gamma_n) &= -\int_D (U_n\cdot\nabla)U_n\cdot V_j\,dx - \int_D \rho(\theta_n)g\cdot V_j\,dx \\
&+\mu_0\int_D (U_n\cdot\nabla)H_n\cdot V_j\,dx + \frac{\mu_0}{2}\int_D M_n\times H_n\cdot\operatorname{curl}V_j\,dx \\
Z_{nj}^2(t,\gamma_n) &= -\int_D (U_n\cdot\nabla)M_n\cdot W_j\,dx - \frac{1}{\delta}\int_D (M_n-\chi_0H_n)\cdot W_j\,dx \\
&+\frac{1}{2}\int_D \operatorname{curl}U_n\times H_n\cdot W_j\,dx - \beta_0\int_D M_n\times H_n\cdot M_n\times W_j\,dx \\
Z_{nj}^3(t,\gamma_n) &= \int_D \theta_nU_n\cdot\nabla v_j\,dx - \nu\int_D |\nabla\theta_n|^2\nabla\theta_n\cdot\nabla v_i\;dx - \int_D \operatorname{div}Q_n\,v_j\,dx \\
Z_{nj}^4(t,\gamma_n) &= -\frac{\tau}{2}\int_D \operatorname{curl}U_n\times Q_n\cdot\Phi_j\,dx + \int_D \mathcal{K}(\theta_n)\operatorname{div}\Phi_j\,dx
\end{aligned}$$

for $1 \le j \le n$.
Notice that Z_n has the same regularity in the time variable t as the function F appearing in the magnetostatic equation and it is continuous and locally lipschitz continuous with respect to the variable γ_n. Hence there exists a unique maximal solution γ_n of (60) defined on a time interval $[0;T_n]$ satisfying $\gamma_n \in H^1(0,T_n;(\mathbb{R}^n)^4)$. We shall prove that $T_n = T$ with the following estimate.
Let $(U_n, M_n, \theta_n, Q_n)$ be the solution of $(\mathcal{P}_\nu^n)$ defined on $(0,T_n)$. We want to verify that

$$\sup_{t\in[0;T_n]} (\|U_n\|^2 + \|M_n\|^2 + \|\theta_n\|_{L^4(D)}^4 + \|Q_n\|^2)(t) < \infty. \tag{61}$$

We multiply equation (58) by b_j and add these equations for $1 \le j \le n$, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\|Q_n\|^2 + \tau\gamma(\|\operatorname{curl}Q_n\|^2 + \|\operatorname{div}Q_n\|^2) + \|Q_n\|^2 = \int_D \mathcal{K}(\theta_n)\operatorname{div}Q_n\,dx. \tag{62}$$

We use the equation (57) that we multiply by $\Theta_j(t) = \int_D \mathcal{K}(\theta_n)\cdot vj\;dx$ and add the equalities for $1 \le j \le n$ to obtain

$$\begin{aligned}
\frac{d}{dt}(\frac{\kappa}{2}\|\theta_n\|^2 + \frac{\alpha}{4}\|\theta_n\|_{L^4(D)}^4) + \nu(\kappa\|\nabla\theta_n\|_{L^4(D)}^4 + 3\alpha\int_D \theta_n^2|\nabla\theta_n|^4\;dx) = \\
-\int_D \operatorname{div}Q_n\,\mathcal{K}(\theta_n)\,dx.
\end{aligned} \tag{63}$$

Adding (62) and (63) lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\kappa\|\theta_n\|^2 + \frac{\alpha}{2}\|\theta_n\|_{L^4(D)}^4 + \tau\|Q_n\|^2) + \nu\kappa\|\nabla\theta_n\|_{L^4(D)}^4 \\
&+3\nu\alpha\|\theta_n|\nabla\theta_n|^2\|^2 + \tau\gamma(\|\operatorname{curl}Q_n\|^2 + \|\operatorname{div}Q_n\|^2) + \|Q_n\|^2 = 0
\end{aligned} \tag{64}$$

Therefore, integrating between 0 and t and using (59), we easily deduce that

$$\begin{aligned}
&(\kappa\|\theta_n\|^2 + \frac{\alpha}{2}\|\theta_n\|_{L^4(D)}^4 + \tau\|Q_n\|^2)(t) + 2\nu\int_0^t \kappa\|\nabla\theta_n\|_{L^4(D)}^4\,ds \\
&+2\int_0^t (3\nu\alpha\|\theta_n|\nabla\theta_n|^2\|^2 + \tau\gamma(\|\operatorname{curl}Q_n\|^2 + \|\operatorname{div}Q_n\|^2) + \|Q_n\|^2)\,ds = \\
&\kappa\|\theta_{0n}^\nu\|^2 + \frac{\alpha}{2}\|\theta_{0n}^\nu\|_{L^4(D)}^4 + \tau\|Q_{0n}\|^2 \le C
\end{aligned} \tag{65}$$

with C independent of n . Similarly, we obtain from equations (55) and (56)

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|U_n\|^2+\eta\|\nabla U_n\|^2=-\int_D\rho(\theta_n)g\cdot U_n\,dx\\ -\mu_0\int_D(U_n\cdot\nabla)M_n\cdot H_n\,dx+\frac{\mu_0}{2}\int_D(M_n\times H_n)\cdot\operatorname{curl}U_n\,dx\end{gathered}\tag{66}$$

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|M_n\|^2+\sigma\|\operatorname{curl}M_n\|^2+\sigma\|\operatorname{div}M_n\|^2+\frac{1}{\delta}\|M_n\|^2=\\ +\frac{\chi_0}{\delta}\int_D H_n\cdot M_n\,dx+\frac{1}{2}\int_D\operatorname{curl}U_n\times H_n\cdot M_n\,dx\end{gathered}\tag{67}$$

so (66) and (67) lead to

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}(\|U_n\|^2+\mu_0\|M_n\|^2)+\eta\|\nabla U_n\|^2+\mu_0\sigma(\|\operatorname{curl}M_n\|^2+\|\operatorname{div}M_n\|^2)+\frac{\mu_0}{\delta}\|M_n\|^2\\ =-\int_D\rho(\theta_n)g\cdot U_n\,dx-\mu_0\int_D(U_n\cdot\nabla)M_n\cdot H_n\,dx+\frac{\mu_0\chi_0}{\delta}\int_D H_n\cdot M_n\,dx.\end{gathered}$$

Using equation (24) for unknown φ_n and data M_n, and testing with $\psi=\varphi_n$, we get

$$\int_D H_n\cdot\partial_t M_n\,dx=-\frac{1}{2}\frac{d}{dt}\|H_n\|^2-\int_D\partial_t F\varphi_n\,dx.$$

Now we multiply equation (56) by $h_j(t)=\displaystyle\int_D H_n\cdot W_j\,dx$ and add the equalities for $1\le j\le n$ to obtain

$$\begin{gathered}\int_D\partial_t M_n\cdot H_n\,dx+\int_D(U_n\cdot\nabla)M_n\cdot H_n\,dx+\sigma\int_D\operatorname{div}M_n\operatorname{div}H_n\,dx\\ +\frac{1}{\delta}\int_D(M_n-\chi_0H_n)\cdot H_n\,dx=-\beta_0\|M_n\times H_n\|^2\end{gathered}\tag{68}$$

so

$$\begin{gathered}\int_D(U_n\cdot\nabla)M_n\cdot H_n\,dx=\frac{1}{2}\frac{d}{dt}\|H_n\|^2+\int_D\partial_t F\varphi_n\,dx-\frac{1}{\delta}\int_D M_n\cdot H_n\,dx\\ -\sigma\int_D\operatorname{div}M_n\,(F-\operatorname{div}M_n)\,dx+\frac{\chi_0}{\delta}\|H_n\|^2-\beta_0\|M_n\times H_n\|^2.\end{gathered}\tag{69}$$

>From (21), we see that

$$\int_D H_n\cdot M_n\,dx=-\|H_n\|^2-\int_D F\varphi_n\,dx$$

therefore

$$\begin{gathered}\int_D(U_n\cdot\nabla)M_n\cdot H_n\,dx=\frac{1}{2}\frac{d}{dt}\|H_n\|^2+\int_D(\partial_t F+\frac{1}{\delta}F)\varphi_n\,dx+\sigma\|\operatorname{div}M_n\|^2\\ -\sigma\int_D\operatorname{div}M_n\,F\,dx+\frac{1+\chi_0}{\delta}\|H_n\|^2-\beta_0\|M_n\times H_n\|^2.\end{gathered}\tag{70}$$

so integrating between 0 and t, we get

$$\begin{gathered}\frac{1}{2}(\|U_n\|^2+\mu_0(\|M_n\|^2+\|H_n\|^2)(t)+\eta\int_0^t\|\nabla U_n\|^2 ds\\ +\int_0^t(\mu_0\sigma(\|\operatorname{curl} M_n\|^2+2\|\operatorname{div} M_n\|^2)+\frac{\mu_0}{\delta}\|M_n\|^2)ds\\ +\int_0^t\frac{\mu_0(1+2\chi_0)}{\delta}\|H_n\|^2\ ds+\beta_0\mu_0\int_0^t\|M_n\times H_n\|^2\ ds=\\ \frac{1}{2}(\|U_{0n}\|^2+\mu_0(\|M_{0n}\|^2+\|H_{0n}\|^2)+\\ \int_0^t\int_D\rho(\theta_n)g\cdot U_n\,dxds-\frac{\mu_0(1+\chi_0)}{\delta}\int_0^t\int_D F\varphi_n\ dxds\\ -\mu_0\int_0^t\int_D\partial_t F\varphi_n\ dxds+\mu_0\sigma\int_0^t\int_D F\operatorname{div} M_n\ dxds\end{gathered}\tag{71}$$

where

$$H_{0n}=\nabla\varphi_{0n},\quad \varphi_{0n}=\mathcal{H}(M_{0n},F(0)).\tag{72}$$

Using the inequalities

$$|\int_0^t\int_D\rho(\theta_n)g\cdot U_n\,dxds|\le C_T+C\int_0^t\|\theta_n\|^2\ ds+\frac{1}{2}\int_0^t\|U_n\|^2\ ds,$$

$$\begin{gathered}\frac{\mu_0(1+\chi_0)}{\delta}|\int_0^t\int_D F\varphi_n\ dxds|+\mu_0|\int_0^t\int_D\partial_t F\varphi_n\ dxds|\le\\ C(\|F\|^2_{L^2(D_T)}+\|\partial_t F\|^2_{L^2(D_T)})+\frac{\mu_0(1+2\chi_0)}{2\delta}\int_0^t\|H_n\|^2\ ds,\end{gathered}$$

$$\mu_0\sigma|\int_0^t\int_D F\operatorname{div} M_n\ dxds|\le C\|F\|^2_{L^2(D_T)}+\mu_0\sigma\int_0^t\|\operatorname{div} M_n\|^2\ ds.$$

We get

$$\begin{gathered}\frac{1}{2}(\|U_n\|^2+\mu_0(\|M_n\|^2+\|H_n\|^2)(t)+\\ \int_0^t[\eta\|\nabla U_n\|^2+\mu_0\sigma(\|\operatorname{curl} M_n\|^2+2\|\operatorname{div} M_n\|^2)+\frac{\mu_0}{\delta}\|M_n\|^2]\ ds\\ +\int_0^t[\frac{\mu_0(1+2\chi_0)}{2\delta}\|H_n\|^2+\beta_0\mu_0\|M_n\times H_n\|^2]\ ds\le\\ A_n+C_T+C\int_0^t\|\theta_n\|^2 ds+\frac{1}{2}\int_0^t\|U_n\|^2\ ds\end{gathered}\tag{73}$$

where

$$A_n=\frac{1}{2}(\|U_{0n}\|^2+\mu_0(\|M_{0n}\|^2+\|H_{0n}\|^2)\le C$$

with C independent of n in view of (59), (72) and (19). Thus thanks to (65) and Gronwall inequality, we deduce that

$$\|U_n(t)\|^2 + \|M_n(t)\|^2 + \|H_n(t)\|^2 \leq C + \exp(Ct) \tag{74}$$

then

$$\begin{gathered}\frac{1}{2}(\|U_n\|^2 + \mu_0(\|M_n\|^2 + \|H_n\|^2)(t)+ \\ \int_0^t [\eta\|\nabla U_n\|^2 + \mu_0\sigma(\|\mathrm{curl}\, M_n\|^2 + 2\|\mathrm{div}\, M_n\|^2) + \frac{\mu_0}{\delta}\|M_n\|^2]\, ds \\ + \int_0^t [\frac{\mu_0(1+2\chi_0)}{2\delta}\|H_n\|^2 + \beta_0\mu_0\|M_n \times H_n\|^2]\, ds \leq C + \exp(Ct).\end{gathered} \tag{75}$$

This ends the proof of (61) so we conclude that $T_n = T$ for all $n \geq 1$.

3.3 Convergence of the Faedo-Galerkine scheme

Let ν be fixed, the estimates (65) and (75) show that

Lemma 1

- *$(U_n)_n$ is uniformly bounded in $L^\infty(0,T;\mathcal{U}_0) \cap L^2(0,T;\mathcal{U})$*
- *$(M_n)_n$ and H_n are uniformly bounded in $L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1_t(D))$*
- *$(M_n \times H_n)_n$ is uniformly bounded in $L^2(0,T;\mathbb{L}^2(D))$*
- *$(Q_n)_n$ is uniformly bounded in $L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1_n(D))$*
- *$(\theta_n)_n$ is uniformly bounded in $L^\infty(0,T;L^4(D)) \cap L^4(0,T;W^{1,4}(D))$.*

Notice that we get the uniform bound of $(H_n)_n$ in $L^2(0,T;\mathbb{H}^1_t(D))$ using the bound of $(M_n)_n$ and (23). Hence we get the convergence

Lemma 2 *Let $\nu > 0$ be fixed. There exists subsequences still denoted (U_n), (M_n), (H_n), (Q_n) and (θ_n) such that when $n \to \infty$*

$$U_n \rightharpoonup U_\nu \quad \textit{weakly} - \star \textit{ in } L^\infty(0,T;\mathbb{L}^2(D)) \textit{and weakly in } \; L^2(0,T;\mathcal{U})$$

$$M_n \rightharpoonup M_\nu,\; H_n \rightharpoonup H_\nu \textit{ weakly} - \star \textit{ in } L^\infty(0,T;\mathbb{L}^2(D)) \textit{ and weakly in } L^2(0,T;\mathbb{H}^1_t(D))$$

$$Q_n \rightharpoonup Q_\nu \textit{ weakly} - \star \textit{ in } L^\infty(0,T;\mathbb{L}^2(D)) \textit{ and weakly in } L^2(0,T;\mathbb{H}^1_n(D))$$

$$\theta_n \rightharpoonup \theta_\nu \textit{ weakly} - \star \textit{ in } L^\infty(0,T;L^4(D)) \textit{ and weakly in } L^4(0,T;W^{1,4}(D))$$

Moreover, we have

$$|\nabla\theta_n|^2\nabla\theta_n \rightharpoonup \Lambda_\nu \textit{ weakly in } L^{\frac{4}{3}}(D_T). \tag{76}$$

In order to pass to the limit in the nonlinear terms, we need strong convergence for the sequences in some spaces. To apply compactness results, we need to estimate the time derivatives of the solutions.

Let us begin with $(\partial_t\theta_n)_n$. We multiply equation (57) by $a_j'(t)$ and add the resulting equalities for $1 \leq j \leq n$ to get

$$\|\partial_t\theta_n\|^2 + \int_D \partial_t\theta_n\nabla\theta_n\cdot U_n\, dx + \frac{\nu}{4}\frac{d}{dt}\|\nabla\theta_n\|^4_{L^4(D)} = -\int_D \mathrm{div}\, Q_n\partial_t\theta_n \tag{77}$$

Since we have $|\int_D \partial_t\theta_n \nabla\theta_n \cdot U_n\,dx| \le \frac{1}{4}\|\partial_t\theta_n\|^2 + \|\nabla\theta_n\|^2_{L^4(D)}\|U_n\|^2_{L^4(D)}$ we obtain

$$\|\partial_t\theta_n\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla\theta_n\|^4_{L^4(D)} \le 2\|\operatorname{div} Q_n\|^2 + 2\|\nabla\theta_n\|^2_{L^4(D)}\|U_n\|^2_{L^4(D)} \tag{78}$$

Therefore, integrating between 0 and t, using (59) and (65), we easily deduce that

$$\begin{aligned}\int_0^t \|\partial_t\theta_n\|^2\,ds + \frac{\nu}{2}\|\nabla\theta_n\|^4_{L^4(D)} &\le \frac{\nu}{2}\|\nabla\theta^\nu_{0n}\|^4_{L^4(D)} + \\ 2\int_0^T \|\operatorname{div} Q_n\|^2\,ds &+ 2\int_0^t \|\nabla\theta_n\|^2_{L^4(D)}\|U_n\|^2_{L^4(D)}\,ds \\ &\le C + 2\int_0^t \|\nabla\theta_n\|^2_{L^4(D)}\|U_n\|^2_{L^4(D)}\,ds\end{aligned} \tag{79}$$

with C independent of n.
Setting $y(t) = \|\nabla\theta_n(t)\|^4_{L^4(D)}$ and $F(t) = \|U_n\|^2_{L^4(D)}$ then from (79), $y(t)$ satisfies the integral inequality

$$y(t) \le C_\nu + 2M_\nu \int_0^t \sqrt{y(s)}\,F(s)\,ds.$$

Using the Gronwall-Bellman-Bihari inequality (see [3]) we deduce

$$y(t) \le \Big(\sqrt{C_\nu} + M_\nu \int_0^t F(s)\,ds\Big)^2.$$

Hence we get for all $t \in [0,T]$ the estimate

$$\|\nabla\theta_n(t)\|^2_{L^4(D)} \le \sqrt{C_\nu} + M_\nu \int_0^t \|U_n(s)\|^2_{L^4(D)}\,ds$$

which leads to

$$\int_0^T \|\partial_t\theta_n\|^2\,ds \le C + 2\sqrt{C_\nu}\int_0^T \|U_n(s)\|^2_{L^4(D)}\,ds + 2M_\nu\Big(\int_0^T \|U_n(s)\|^2_{L^4(D)}\,ds\Big)^2 \tag{80}$$

and we conclude that $(\partial_t\theta_n)_n$ is uniformly bounded in $L^2(D_T)$ with respect to n.
To estimate $\partial_t U_n$, $\partial_t M_n$ and $\partial_t Q_n$ we need some notations. For a function f defined on $[0,T]$ with values in a space V, let $\widetilde{f}$ be the function equal to f on $[0,T]$ and to 0 elsewhere and let $\widehat{f}$ be its Fourier transform defined by

$$\widehat{f}(\tau) = \int_{\mathbb{R}} \exp(-2i\pi t\tau)\widetilde{f}(t)\,dt = \int_0^T \exp(-2i\pi t\tau)f(t)\,dt, \quad \tau \in \mathbb{R}.$$

We will prove that for $0 < \gamma < 1/4$,

$$\int_{\mathbb{R}} |\tau|^{2\gamma}\|\widehat{U}_n(\tau)\|^2 \; d\tau \le C. \tag{81}$$

Proceeding as in [22] (see also [16]) and since $(U_n)_n$ is uniformly bounded in $L^2(0,T;\mathcal{U})$, it is enough to verify that

$$|\tau|\,\|\widehat{U}_n(\tau)\|^2 \le C\|\widehat{U}_n(\tau)\|_{\mathcal{U}} + C\|\widehat{U}_n(\tau)\|, \quad \forall\tau \in \mathbb{R}. \tag{82}$$

We write the equation (55) of U_n in the form

$$\frac{d}{dt}\int_D U_n \cdot V_j \, dx = \langle \mathcal{L}_n, V_j \rangle, \quad U_n(0) = U_{0n} \tag{83}$$

for all $1 \leq j \leq n$ where the linear form $\mathcal{L}_n$ is defined on $\mathcal{U}$ by

$$\begin{aligned} \langle \mathcal{L}_n, \varphi \rangle = \int_D (U_n \cdot \nabla) U_n \cdot \varphi \, dx + \eta \int_D \nabla U_n \cdot \nabla \varphi \, dx + \int_D \rho(\theta_n) g \cdot \varphi \, dx \\ -\mu_0 \int_D (M_n \cdot \nabla) H_n \cdot \varphi \, dx - \frac{\mu_0}{2} \int_D M_n \times H_n \cdot \operatorname{curl} \varphi \, dx \end{aligned} \tag{84}$$

We have $\mathcal{L}_n \in \mathcal{U}'$ p.p. $t \in (0, T)$ and

$$\|\mathcal{L}_n\|_{\mathcal{U}'} \leq C(\|U_n\|^2_{H^1(D)} + \|U_n\|_{H^1(D)} + \|\rho(\theta_n)\| +$$
$$\|M_n\|_{H^1(D)} \|H_n\|_{H^1(D)} + \|M_n \times H_n\|)$$

and we conclude thanks to Lemma 1 that $(\mathcal{L}_n)$ is uniformly bounded in $L^1(0, T; \mathcal{U}')$. Now we rewrite (99) as follows

$$\frac{d}{dt}\int_D \widetilde{U}_n \cdot V_j \, dx = \langle \widetilde{\mathcal{L}_n}, V_j \rangle + \Big(\int_D U_{0n} \cdot V_j \ dx\Big)\delta_0 - \Big(\int_D U_n(T) \cdot V_j \ dx\Big)\delta_T \tag{85}$$

for $1 \leq j \leq n$ where δ_a denotes the Dirac distribution at $a \in \mathbb{R}$. Therefore, we obtain

$$2i\pi\tau \int_D \widehat{U}_n \cdot V_j \, dx = \langle \widehat{\mathcal{L}_n}, v_j \rangle + \int_D U_{0n} \cdot V_j \ dx - \exp(-2i\pi T\tau) \int_D U_n(T) \cdot V_j \ dx \tag{86}$$

Next we multiply equality (86) by $\overline{\widehat{\alpha_j}}(\tau)$ the conjugate of $\widehat{\alpha_j}(\tau)$ and add the equalities for $1 \leq j \leq n$ to get

$$2i\pi\tau \|\widehat{U}_n\|^2 = \langle \widehat{\mathcal{L}_n}, \overline{\widehat{U}_n} \rangle + \int_D U_{0n} \cdot \overline{\widehat{U}_n} \ dx - \exp(-2i\pi T\tau) \int_D U_n(T) \cdot \overline{\widehat{U}_n} \ dx \tag{87}$$

therefore since for all $\tau \in \mathbb{R}$, we have

$$\|\widehat{\mathcal{L}_n}(\tau)\|_{\mathcal{U}'} \leq \int_0^T \|\mathcal{L}_n(t)\|_{\mathcal{U}'} \, dt \leq C$$

then using Plancherel identity we get (82).
Similar proofs work for $\partial_t M_n$ and $\partial_t Q_n$. The above results are summarized in

Lemma 3 *There exists $C_\nu > 0$ such that for all n*

$$\int_{\mathbb{R}} |\tau|^{2\gamma} (\|\widehat{U}_n(\tau)\|^2 + \|\widehat{M}_n(\tau)\|^2 + \|\widehat{Q}_n(\tau)\|^2) \ d\tau \leq C_\nu \tag{88}$$

Moreover we have

$$\|\partial_t \theta_n\|_{L^2(D_T)} \leq C_\nu \tag{89}$$

Combining the bounds of Lemma 1 and Lemma 3 and applying Lions compactness lemma for (U_n, M_n, Q_n) and and Aubin compactness lemma for θ_n we get the strong convergence results we get the strong convergence results

Lemma 4 *For $\nu > 0$ fixed, we have*

$$(U_n,\ M_n,\ H_n,\ \theta_n,\ Q_n) \to (U_\nu,\ M_\nu,\ H_\nu,\ \theta_\nu,\ Q_\nu) \text{ strongly in } (\mathbb{L}^2(D_T))^5 \tag{90}$$

Moreover we have $H_\nu = \mathcal{H}(M_\nu, F)$.

The strong convergence of $(H_n)_n$ is a consequence of the continuity of operator $\mathcal{H}$ (see subsection 1.3).

Thanks to Lemma 2 and Lemma 4, we can pass to the limit in problem $(\mathcal{P}_\nu^n)$ when $n \to \infty$. We get that $(U_\nu, M_\nu, H_\nu, \theta_\nu, Q_\nu)$ satisfies the equations of system $(\mathcal{P}_\nu)$ except for the temperature equation satisfied by θ_ν for which we obtain

$$\frac{d}{dt}\int_D \theta_\nu a\, dx - \int_D \theta_\nu U_\nu \cdot \nabla a\, dx + \nu \int_D \Lambda_\nu \cdot \nabla a\, dx = -\int_D \operatorname{div} Q_\nu\, a \tag{91}$$

for all $a \in W^{1,4}(D)$ and Λ_ν being defined in (76).

Passing to the limit in the temperature equation. Hereafter, we detail the procedure of passing to the limit in the equation of θ_n. First we introduce some notations. Let $W = W^{1,4}(D)$ and A the nonlinear operator defined on W by

$$\langle A(\varphi), \psi\rangle = \int_D |\nabla\varphi|^2 \nabla\varphi \cdot \nabla\psi\ dx, \quad \forall \varphi, \psi \in W \tag{92}$$

then $A(\varphi) \in W'$ for all $\varphi \in W$ and

$$\|A(\varphi)\|_{W'} \leq \|\nabla\varphi\|^3_{L^4(D)}. \tag{93}$$

Next we define on $\mathbb{L}^2(D) \times L^4(D)$ the bilinear operator B by

$$\langle B(\xi, \varphi), \psi\rangle = \int_D \varphi\ \xi \cdot \nabla\psi\ dx, \quad \forall\ \xi \in \mathbb{L}^2(D), \varphi \in L^4(D), \psi \in W.$$

It holds that $B(\xi, \varphi) \in W'$ for all $(\xi, \varphi) \in \mathbb{L}^2(D) \times L^4(D)$ and

$$\|B(\xi, \varphi)\|_{W'} \leq \|\varphi\|_{L^4(D)}\|\xi\| \tag{94}$$

and we have $L^2(D) \subset W'$ with

$$\|f\|_{W'} \leq (\text{mes } (D))^{1/4}\|f\|, \quad \forall f \in L^2(D). \tag{95}$$

We multiply equation (57) by a function $f \in \mathcal{C}^1([0,T])$ such that $f(T) = 0$ and integrate by parts. Then integrating by parts, we get for all $1 \leq j \leq n$

$$\begin{aligned} &-\int_{D_T} f'(t)\theta_n v_j\, dxdt - f(0)\int_D \theta_{0n}^\nu v_j\, dx - \int_{D_T} f(t)\theta_n U_n \cdot \nabla v_j\, dxdt \\ &+\nu \int_{D_T} f(t)|\nabla\theta_n|^2 \nabla\theta_n \cdot \nabla v_j\, dxdt = -\int_{D_T} f(t) \operatorname{div} Q_n\, v_j\, dxdt. \end{aligned} \tag{96}$$

Therefore letting $j \geq 1$ be fixed and $n \to \infty$, using the previous convergence together with the fact that $\theta_{0n}^\nu \to \theta_0^\nu$ strongly in $L^2(D)$, we obtain for all $j \geq 1$

$$\begin{aligned} &-\int_{D_T} f'(t)\theta_\nu v_j\, dxdt - f(0)\int_D \theta_0^\nu v_j\, dx - \int_{D_T} f(t)\theta_\nu U_\nu \cdot \nabla v_j\, dxdt + \\ &\nu \int_{D_T} f(t)\Lambda_\nu \cdot \nabla v_j\, dxdt = -\int_{D_T} f(t) \operatorname{div} Q_\nu\, v_j\, dxdt \end{aligned} \tag{97}$$

Henceforth it holds for all $a \in W^{1,4}(D)$ and $f \in \mathcal{C}^1([0,T])$ such that $f(T) = 0$ the following identity

$$\begin{aligned} -\int_{D_T} f'(t)\theta_\nu a\, dxdt - f(0)\int_D \theta_0^\nu a\, dx - \int_{D_T} f(t)\theta_\nu U_\nu \cdot \nabla a\, dxdt \\ +\nu\int_{D_T} f(t)\Lambda_\nu \cdot \nabla a\, dxdt = -\int_{D_T} f(t)\mathrm{div}\, Q_\nu\, a\, dxdt \end{aligned} \tag{98}$$

In particular for $f \in \mathcal{D}(]0,T[)$, we obtain (91) which is satisfied in $\mathcal{D}'(]0,T[)$. Thus we get in $\mathcal{D}\prime(D)$ and p.p. $t \in (0,T)$

$$\partial_t\theta_\nu - B(U_\nu, \theta_\nu) + \nu\Lambda_\nu = -\mathrm{div}\, Q_\nu \tag{99}$$

and we deduce that $\partial_t\theta_\nu \in L^{\frac{4}{3}}(0,T;(W^{1,4}(D))')$. Since $\theta_\nu \in L^4(0,T;W^{1,4}(D))$ then $\theta_\nu \in \mathcal{C}([0,T];L^4(D))$ so that $\theta_\nu(0)$ is well defined and multiplying equation (91) by a function $f \in \mathcal{C}^1([0,T])$ such that $f(T) = 0$ and integrating by parts, we obtain the equation

$$\begin{aligned} -\int_{D_T} f'(t)\theta_\nu a\, dxdt - f(0)\int_D \theta_\nu(0) a\, dx - \int_{D_T} f(t)\theta_\nu U_\nu \cdot \nabla a\, dxdt \\ +\nu\int_{D_T} f(t)\Lambda_\nu \cdot \nabla a\, dxdt = -\int_{D_T} f(t)\mathrm{div}\, Q_\nu\, a\, dxdt \end{aligned} \tag{100}$$

for all $a \in W^{1,4}(D)$, thus comparing (98) and (100) we deduce that

$$\theta_\nu(0) = \theta_0^\nu. \tag{101}$$

Next, since $(\theta_n(T))_n$ is bounded in $L^4(D)$, then at least for a subsequence, $(\theta_n(T))_n$ converge weakly in $L^4(D)$. Proceeding as previously using test functions $f \in \mathcal{C}^1([0,T])$ (with possibly the condition $f(0) = 0$ for simplicity), we prove that

Lemma 5

$$\theta_n(T) \rightharpoonup \theta_\nu(T) \textit{ weakly in } L^4(D).$$

It remains to prove the

Lemma 6 *Let Λ_ν be the weak limit of $(|\nabla\theta_n|^2\nabla\theta_n)$ in $\mathbb{L}^{4/3}(D_T)$. We have*

$$\Lambda_\nu = |\nabla\theta_\nu|^2\nabla\theta_\nu$$

Proof. We use the monotonicity of the operator A defined in (92). We set

$$A_n(\varphi) = \int_0^T \langle A(\theta_n) - A(\varphi), \theta_n - \varphi\rangle\, dt \geq 0, \quad \forall\varphi \in L^4(0,T;W^{1,4}(D)).$$

We multiply equation (57) by a_j and add these equations for $1 \leq j \leq n$, then integrating between 0 and T, we obtain

$$\frac{1}{2}\|\theta_n(T)\|^2 + \nu\int_0^T \|\nabla\theta_n\|^4_{L^4(D)}\, ds = \frac{1}{2}\|\theta_{0n}^\nu\|^2 - \int_{D_T} \mathrm{div}\, Q_n\theta_n\, dxdt \tag{102}$$

We write

$$A_n(\varphi) = -\int_{D_T} |\nabla\theta_n|^2 \nabla\theta_n \cdot \nabla\varphi \; dxdt - \int_0^T \langle A(\varphi), \theta_n - \varphi\rangle \; dt$$
$$+\frac{1}{\nu}\Big[\frac{1}{2}(\|\theta_{0n}^\nu\|^2 - \|\theta_n(T)\|^2 \; dx) - \int_{D_T} \operatorname{div} Q_n \theta_n \; dxdt\Big].$$

Therefore by the convergence results given before, we get

$$\begin{aligned}\limsup A_n(\varphi) \leq -\int_{D_T} \Lambda_\nu \cdot \nabla\varphi \; dxdt - \int_0^T \langle A(\varphi), \theta_\nu - \varphi\rangle \; dt \\ +\frac{1}{\nu}\Big[\frac{1}{2}(\|\theta_0^\nu\|^2 - \|\theta_\nu(T)\|^2 \; dx) - \int_{D_T} \operatorname{div} Q_\nu \theta_\nu \; dxdt\Big].\end{aligned} \tag{103}$$

Taking $a = \theta_\nu$ in equation (91) and integrating with respect to the time variable, we get

$$\frac{1}{2}(\|\theta_\nu(T)\|^2 - \|\theta_0^\nu\|^2) + \nu \int_{D_T} \Lambda_\nu \cdot \nabla\theta_\nu \; dxdt = -\int_{D_T} \operatorname{div} Q_\nu \theta_\nu \; dxdt.$$

Coming back to inequality (103), we deduce that

$$\limsup A_n(\varphi) \leq \int_{D_T} \Lambda_\nu \cdot (\nabla\theta_\nu - \nabla\varphi) \; dxdt - \int_0^T \langle A(\varphi), \theta_\nu - \varphi\rangle \; dt. \tag{104}$$

Therefore, for all $\varphi \in L^4(0,T;W^{1,4}(D))$ we have

$$0 \leq \int_{D_T} \Lambda_\nu \cdot (\nabla\theta_\nu - \nabla\varphi) \; dxdt - \int_0^T \langle A(\varphi), \theta_\nu - \varphi\rangle \; dt \tag{105}$$

Taking $\varphi = \theta_\nu - \lambda\psi$ with $\lambda > 0$ and $\psi \in L^4(0,T;W^{1,4}(D))$, we get

$$0 \leq \int_{D_T} \Lambda_\nu \cdot \nabla\psi \; dxdt - \int_0^T \langle A(\theta_\nu - \lambda\psi), \psi\rangle \; dt \tag{106}$$

and since operator A is hemicontinuous, then letting $\lambda \to 0$, we obtain

$$0 \leq \int_{D_T} \Lambda_\nu \cdot \nabla\psi \; dxdt - \int_{D_T} |\nabla\theta_\nu|^2 \nabla\theta_\nu \cdot \nabla\psi \; dxdt \tag{107}$$

for all $\psi \in L^4(0,T;W^{1,4}(D))$ which leads to the result. □

Going along the same lines, one can pass to the limit in the other equations thanks to the strong convergence results of Lemma 4. One deduces that U_ν, M_ν and Q_ν satisfy the weak formulations given in Theorem 2. To verify the corresponding initial conditions, one can proceed as for θ_ν. In order to prove energy estimates (65) and (73), we multiply (46) and (48) by a test function $f \in \mathcal{D}(]0,T[)$ such that $f \geq 0$ and integrate between 0 and T. Thus using Lemma 2, we can take the lim inf which leads to the desired results. Hence we get a global weak solution with finite energy to problem $(\mathcal{P}_\nu)$. This ends proof of Theorem 2.

4 Proof of Theorem 1

4.1 An auxiliary problem

For $\nu > 0$, let $(U_\nu,\ M_\nu,\ H_\nu,\ \theta_\nu,\ Q_\nu)$ be the solution of problem $(\mathcal{P}_\nu)$ provided by Theorem 2. In order to get some compacity result to deal with the limit of the nonlinear terms of the problem $(\mathcal{P}_\nu)$ when $\nu \to 0$, we introduce the following auxiliary function

$$\zeta_\nu = \tau\gamma \mathrm{div}\, Q_\nu - \mathcal{K}(\theta_\nu). \tag{108}$$

Taking the divergence of the heat-flux equation in (45), we easily see that ζ_ν satisfies the equation

$$\partial_t \zeta_\nu - \gamma \Delta \zeta_\nu = -\gamma \mathrm{div}\, Q_\nu - \frac{\tau\gamma}{2}\mathrm{div}\,(\mathrm{curl}\, U_\nu \times Q_\nu) - \partial_t \mathcal{K}(\theta_\nu). \tag{109}$$

Multiplying the temperature equation in (45) by $\mathcal{K}'(\theta_\nu)$ we get

$$\partial_t \mathcal{K}(\theta_\nu) + U_\nu \cdot \nabla \mathcal{K}(\theta_\nu) - \nu \mathcal{K}'(\theta_\nu) \nabla \cdot (|\nabla\theta_\nu|^2 \nabla\theta_\nu) = -\mathcal{K}'(\theta_\nu)\,\mathrm{div}\, Q_\nu. \tag{110}$$

Hence ζ_ν satisfies the auxiliary problem

$$\begin{aligned} &\partial_t \zeta_\nu - \gamma\Delta\zeta_\nu = g_\nu + \mathrm{div}\, G_\nu + \mu_\nu \text{ in } D_T \\ &\zeta_\nu(0) = \zeta_0^\nu \text{ in } D,\ \zeta_\nu = 0 \text{ on } \Gamma_T \end{aligned} \tag{111}$$

where $\zeta_0^\nu = \tau\gamma \mathrm{div}\, Q_0 - \mathcal{K}(\theta_0^\nu)$ and

$$\begin{aligned} g_\nu &= (\kappa - \gamma)\mathrm{div}\, Q_\nu \\ G_\nu &= U_\nu \mathcal{K}(\theta_\nu) - \nu\kappa |\nabla\theta_\nu|^2 \nabla\theta_\nu - 3\alpha\nu\theta_\nu^2 |\nabla\theta_\nu|^2 \nabla\theta_\nu - \frac{\tau\gamma}{2}\mathrm{curl}\, U_\nu \times Q_\nu \\ \mu_\nu &= 6\nu\alpha\theta_\nu |\nabla\theta_\nu|^4 + 3\alpha\theta_\nu^2 \mathrm{div}\, Q_\nu. \end{aligned} \tag{112}$$

We note that ζ_0^ν converge strongly in $L^{\frac{4}{3}}(D)$ towards $\zeta_0 = \tau\gamma \mathrm{div}\, Q_0 - \mathcal{K}(\theta_0)$ and we have the following results

Lemma 7 *There exists $C > 0$ which is independent of ν such that*

$$\begin{aligned} &\|g_\nu\|_{L^2(0,T;\ L^2(D))} \le C \\ &\|G_\nu\|_{L^{4/3}(0,T;\ \mathbb{L}^{12/11}(D))} \le C \\ &\|\mu_\nu\|_{L^1(0,T;\ L^1(D))} \le C \end{aligned} \tag{113}$$

Proof. We use the energy estimates (46) and (48) satisfied by $(U_\nu,\ M_\nu,\ H_\nu,\ \theta_\nu,\ Q_\nu)$. So clearly g_ν is bounded in $L^2(0,T;L^2(D))$. We write G_ν in the form $G_\nu = G_{1,\nu} - G_{2,\nu} - G_{3,\nu} - G_{4,\nu}$ with

$$G_{1,\nu} = U_\nu \mathcal{K}(\theta_\nu),\ G_{2,\nu} = \nu\kappa|\nabla\theta_\nu|^2\nabla\theta_\nu,$$

$$G_{3,\nu} = 3\alpha\nu\theta_\nu^2 |\nabla\theta_\nu|^2 \nabla\theta_\nu,\ G_{4,\nu} = \tfrac{\tau\gamma}{2}\mathrm{curl}\, U_\nu \times Q_\nu.$$

Since $L^2(0,T;\mathcal{U})$ is continuously embedded in $L^2(0,T;\mathbb{L}^6(D))$ then we see that U_ν is bounded in $L^2(0,T;\mathbb{L}^6(D))$ with respect to ν and as $\mathcal{K}(\theta_\nu)$ is bounded in $L^\infty(0,T;L^{\frac{4}{3}}(D))$, we deduce that $G_{1,\nu}$ is bounded in $L^2(0,T;\mathbb{L}^{12/11}(D))$. $G_{2,\nu}$ is clearly bounded in $L^{4/3}(0,T;\mathbb{L}^{4/3}(D))$. Next since Q_ν is bounded in $L^\infty(0,T;\mathbb{L}^2(D))\cap L^2(0,T;\mathbb{H}^1(D))$ then using an interpolation result we get that Q_ν is bounded in $L^4(0,T;\mathbb{L}^3(D))$. Therefore as $\operatorname{curl} U_\nu$ is bounded in $L^2(0,T;\mathbb{L}^2(D))$, we conclude that $G_{4,\nu}$ is bounded in $L^{4/3}(0,T;\mathbb{L}^{6/5}(D))$. Now we write $|G_{3,\nu}|=(3\alpha\nu)h_\nu k_\nu$ with

$$h_\nu=|\theta_\nu|^{3/2}|\nabla\theta_\nu|^3\in L^{4/3}(0,T;L^{4/3}(D))$$

$$k_\nu=|\theta_\nu|^{1/2}\in L^\infty(0,T;L^8(D)).$$

So $G_{3,\nu}\in L^{4/3}(0,T;L^{8/7}(D))$ and p.p. $t\in(0,T)$

$$\begin{aligned}\|G_{3,\nu}(t)\|_{L^{8/7}(D)}&\le 3\alpha\nu\|h_\nu(t)\|_{L^{4/3}(D)}\|k_\nu(t)\|_{L^8(D)}\\ &\le 3\alpha\nu\Big(\int_D|\theta_\nu|^2|\nabla\theta_\nu|^4\,dx\Big)^{3/4}\ \|\theta_\nu\|^{1/2}_{L^4(D)}\end{aligned}$$

then

$$\|G_{3,\nu}\|_{L^{4/3}(0,T;L^{8/7}(D))}\le 3\alpha\nu\Big(\int_{D_T}|\theta_\nu|^2|\nabla\theta_\nu|^4\,dxdt\Big)^{3/4}\ \|\theta_\nu\|^{1/2}_{L^\infty(0,T;L^4(D))}$$

therefore $G_{3,\nu}$ is uniformly bounded in $L^{4/3}(0,T;\mathbb{L}^{8/7}(D))$ with respect to ν which implies that G_ν is bounded in $L^{4/3}(0,T;\mathbb{L}^{12/11}(D))$. To deal with μ_ν, we split it into two terms $\mu_\nu=\mu_{1,\nu}+\mu_{2,\nu}$ with $\mu_{1,\nu}=6\nu\alpha\theta_\nu|\nabla\theta_\nu|^4$ and $\mu_{2,\nu}=3\alpha\theta_\nu^2\operatorname{div}Q_\nu$. Then clearly the second term $\mu_{2,\nu}$ is bounded in $L^2(0,T;L^1(D))$ whereas for the first term, we have

$$\begin{aligned}&\int_{D_T}|\mu_{1,\nu}|\,dxdt=6\alpha\nu\int_{D_T}|\theta_\nu||\nabla\theta_\nu|^4\,dxdt\le\\ &6\alpha\nu\int_{\{|\theta_\nu|\le1\}}|\nabla\theta_\nu|^4\ +6\alpha\nu\int_{\{|\theta_\nu|>1\}}|\theta_\nu|^2|\nabla\theta_\nu|^4\,dxdt\le C\end{aligned}\tag{114}$$

The Lemma is then proved. $\square$

To use the known regularity results on solutions of parabolic equations, we split the function ζ_ν into two terms $\zeta_{1,\nu}$ and $\zeta_{2,\nu}$ where $\zeta_{1,\nu}$ satisfies the problem

$$\begin{aligned}&\partial_t\zeta_{1,\nu}-\gamma\Delta\zeta_{1,\nu}=g_\nu+\operatorname{div}G_\nu\text{ in }D_T\\ &\zeta_{1,\nu}(0)=\zeta_0^\nu\text{ in }D,\ \zeta_{1,\nu}=0\text{ on }\Gamma_T\end{aligned}\tag{115}$$

with g_ν and G_ν bounded in $L^{12/11}(0,T;\mathbb{L}^{12/11}(D))$, ζ_0^ν bounded in $L^{\frac43}(D)$ whereas the function $\zeta_{2,\nu}$ verifies the problem

$$\begin{aligned}&\partial_t\zeta_{2,\nu}-\gamma\Delta\zeta_{2,\nu}=\mu_\nu\text{ in }D_T\\ &\zeta_{2,\nu}(0)=0\text{ in }D,\ \zeta_{2,\nu}=0\text{ on }\Gamma_T\end{aligned}\tag{116}$$

with μ_ν bounded in $L^1(0,T;L^1(D))$. We have the result

Lemma 8 *There exists $C>0$ which is independent of ν such that*

$$\begin{aligned}&\|\zeta_{1,\nu}\|_{L^{12/11}(0,T;W^{1,12/11}(D))}+\|\partial_t\zeta_{1,\nu}\|_{L^{12/11}(0,T;W^{-1,12/11}(D))}\le C\\ &\|\zeta_{2,\nu}\|_{L^p(0,T;W^{1,p}(D))}+\|\partial_t\zeta_{2,\nu}\|_{L^p(0,T;W^{-1,p}(D))+L^1(0,T;L^1(D))}\le C\end{aligned}\tag{117}$$

for all $1\le p<\dfrac54$.

Proof. The bounds for $\zeta_{1,\nu}$ result from the classical L^p estimates of solutions of parabolic equations (see for example [9], [12] and the references therein) whereas for $\zeta_{2,\nu}$, we use a result given in [2], see also [4] . The estimate of $\partial_t \zeta_{2,\nu}$ follows from the equation (116). □

Combining the above estimates and using Aubin's compactness lemma we get the convergence results

Lemma 9 *For subsequences we have*

$$\begin{aligned}
&\zeta_{1,\nu} \to \zeta_1 \text{ strongly in } L^{12/11}(0,T;L^{q_1}(D)),\ 1 \leq q_1 < (12/11)^\star = 12/7 \\
&\zeta_{2,\nu} \to \zeta_2 \text{ strongly in } L^{p}(0,T;L^{q_2}(D)),\ 1 \leq q_2 < p^\star = \tfrac{3p}{3-p},\quad 1 \leq p < 5/4 \qquad (118)\\
&\zeta_{\nu} \to \zeta \text{ strongly in } L^{12/11}(0,T;L^{q_1}(D)),\ 1 \leq q_1 < 12/7.
\end{aligned}$$

The strong convergence of ζ_ν is crucial to obtain the limit in the nonlinear term $\mathcal{K}(\theta_\nu)$ since the regularized equation of θ_ν does not provide any uniform bound of the space derivative. We will precise this point in the following subsection.

4.2 Convergence in the temperature equation as $\nu \to 0$

Let $\nu > 0$ and $(U_\nu, M_\nu, H_\nu, \theta_\nu, Q_\nu)$ be the solution of problem $(\mathcal{P}_\nu)$ provided by Theorem 2. From the estimates (46), we deduce that

Lemma 10 *For a subsequence, we have*

$$\theta_\nu \rightharpoonup \theta \text{ weakly -}* \text{ in } L^\infty(0,T;L^4(D)) \qquad (119)$$

$$\nu|\nabla\theta_\nu|^2\nabla\theta_\nu \to 0 \text{ strongly in } \mathbb{L}^{4/3}(D_T).$$

Next, assume momentarily that

$$\begin{aligned}
&U_\nu \to U \text{ strongly in } \mathbb{L}^2(0,T;\mathcal{U}_0) \\
&\operatorname{div} Q_\nu \rightharpoonup \operatorname{div} Q \text{ weakly in } L^2(D_T)
\end{aligned} \qquad (120)$$

then we can perform the limit when $\nu \to 0$ in the equation of θ_ν. We get the result

Lemma 11 $\theta \in L^\infty(0,T;L^4(D))$ *is a weak solution of the transport equation*

$$\begin{aligned}
&\partial_t\theta + U\cdot\nabla\theta = -\operatorname{div} Q \quad in \quad D_T \\
&\theta(0) = \theta_0 \quad in \quad D.
\end{aligned} \qquad (121)$$

Moreover, $\theta \in C([0,T];L^q(D))$ *for all* $q < 4$ *and* $\theta \in C([0,T];L^4(D)$ *weak).*

We recall that $\theta \in L^\infty(0,T;L^4(D))$ is a weak solution of (121) if

$$\int_{D_T} \theta(\partial_t a + U\cdot\nabla a)\,dxdt = \int_{D_T} \operatorname{div} Q\, a\,dxdt - \int_D \theta_0\, a(0)\,dx \qquad (122)$$

for all $a \in \mathcal{D}([0,T[\times\overline{D})$ so equation (121) is satisfied in $\mathcal{D}'(D_T)$. The regularity result given in Lemma 11 is well known in transport equation theory as said in Remark 2.3, which gives a sense to the initial condition in (121). Moreover, we have

Lemma 12

$$\theta_\nu(T) \rightharpoonup \theta(T) \ \textit{weakly in } L^2(D).$$

Proof. $\theta(T)$ is well defined in $L^2(D)$, let v be the weak limit of $\theta_\nu(T)$ in $L^4(D)$ and $L^2(D)$. We use test functions $a \in \mathcal{D}(D)$ in the equation (52) of θ_ν, we multiply by $f \in \mathcal{D}(]0,T])$ and integrating between 0 and T, we get

$$f(T)\int_D \theta_\nu(T)a \; dx = \int_{D_T} f(t)\,\theta_\nu\, U_\nu \cdot \nabla a \; dxdt$$
$$-\nu \int_{D_T} f(t)\,|\nabla\theta_\nu|^2 \nabla\theta_\nu \cdot \nabla a \; dxdt - \int_{D_T} f(t)\,\mathrm{div}\, Q_\nu \, a \; dxdt$$

thus passing to the limit we get

$$f(T)\int_D v\, a \; dx = \int_{D_T} f(t)\,\theta\, U \cdot \nabla a \; dxdt - \int_{D_T} f(t)\,\mathrm{div}\, Q \, a \; dxdt.$$

Using equation (121) of θ, we deduce that for all $a \in \mathcal{D}(D)$, $f \in \mathcal{D}(]0,T])$

$$f(T)\int_D v\, a \; dx = f(T)\int_D \theta(T) a \; dx$$

hence $v = \theta(T)$. □

Consider now the equation of θ_ν from (45) that we rewrite using ζ_ν in the form

$$\begin{aligned} &\partial_t\theta_\nu + U_\nu \cdot \nabla\theta_\nu - \nu\nabla\cdot(|\nabla\theta_\nu|^2\nabla\theta_\nu) + \tfrac{1}{\tau\gamma}\mathcal{K}(\theta_\nu) = -\tfrac{1}{\tau\gamma}\zeta_\nu \text{ in } D_T \\ &\theta_\nu(0) = \theta_0^\nu \text{ in } D, \quad \nu|\nabla\theta_\nu|^2\nabla\theta_\nu\cdot\mathbf{n} = 0 \text{ on } \Gamma_T. \end{aligned} \tag{123}$$

Under the hypothesis (120), we have

Lemma 13 *Let χ be the weak-$*$ limit of $\mathcal{K}(\theta_\nu)$ in $L^\infty(0,T;L^{4/3}(D))$. We have*

$$\chi = \mathcal{K}(\theta).$$

Proof. We use the monotonicity of the function $\mathcal{K}$ and the strong convergence in $L^1(0,T;L^{4/3}(D))$ of ζ_ν given in Lemma 9. We set

$$a_\nu(\varphi) = \int_{D_T} (\mathcal{K}(\theta_\nu) - \mathcal{K}(\varphi))(\theta_\nu - \varphi)\; dxdt \ge 0, \quad \forall \varphi \in L^\infty(0,T;L^4(D)).$$

We multiply equation (123) by θ_ν and integrate over D_T to get

$$\frac{1}{\tau\gamma}\int_{D_T} \mathcal{K}(\theta_\nu)\theta_\nu \; dxdt = -\frac{1}{\tau\gamma}\int_{D_T} \zeta_\nu\theta_\nu \; dxdt - \nu\int_{D_T} |\nabla\theta_\nu|^4 \; dxdt$$
$$-\frac{1}{2}\int_D \theta_\nu^2(T)dx + \frac{1}{2}\int_D (\theta_0^\nu)^2 \; dx.$$

Then

$$a_\nu(\varphi) = -\int_{D_T} \mathcal{K}(\theta_\nu)\varphi \; dxdt - \int_{D_T} \mathcal{K}(\varphi)(\theta_\nu - \varphi)\; dxdt - \int_{D_T} \zeta_\nu\theta_\nu \; dxdt$$
$$-\tau\gamma\nu\int_{D_T} |\nabla\theta_\nu|^4 \; dxdt - \frac{\tau\gamma}{2}\int_D \theta_\nu^2(T)dx + \frac{\tau\gamma}{2}\int_D (\theta_0^\nu)^2 \; dx.$$

>From Lemma 12, we have

$$\liminf \int_D |\theta_\nu(T)|^2 \ dx \geq \int_D |\theta(T)|^2 \ dx$$

therefore using hypothesis (44) we obtain

$$\limsup a_\nu(\varphi) \leq -\int_{D_T} \chi\varphi \ dxdt - \int_{D_T} \mathcal{K}(\varphi)(\theta-\varphi) \ dxdt - \int_{D_T} \zeta\theta \ dxdt \\ -\frac{\tau\gamma}{2}\int_D \theta^2(T)dx + \frac{\tau\gamma}{2}\int_D \theta_0^2 \ dx$$

for all φ in $L^\infty(0,T;L^4(D))$. On the other hand, taking the limit in the weak formulation of problem (123), we deduce that θ satisfies the equation

$$\begin{aligned} &\partial_t\theta + U\cdot\nabla\theta + \tfrac{1}{\tau\gamma}\chi = -\tfrac{1}{\tau\gamma}\zeta \text{ in } D_T \\ &\theta(0) = \theta_0 \text{ in } D. \end{aligned} \tag{124}$$

Multiplying the equation by θ we get

$$-\int_{D_T} \zeta\theta \ dxd - \frac{\tau\gamma}{2}\int_D \theta^2(T)dx + \frac{\tau\gamma}{2}\int_D \theta_0^2 \ dx = \int_{D_T} \chi\theta \ dxdt.$$

Therefore

$$\limsup a_\nu(\varphi) \leq \int_{D_T} \chi(\theta-\varphi) \ dxdt - \int_{D_T} \mathcal{K}(\varphi)(\theta-\varphi) \ dxdt.$$

We deduce that

$$\int_{D_T} (\chi - \mathcal{K}(\varphi))(\theta-\varphi) \ dxdt \geq 0, \quad \forall\varphi \in L^\infty(0,T;L^4(D)). \tag{125}$$

So, taking in (125) $\varphi = \theta - \lambda\psi$ with $\lambda > 0, \psi \in L^\infty(0,T;L^4(D))$, we get

$$\int_{D_T} (\chi - \mathcal{K}(\theta-\lambda\psi))\psi \ dxdt \geq 0, \quad \forall\psi \in L^\infty(0,T;L^4(D)), \forall\lambda > 0 \tag{126}$$

and letting $\lambda \to 0$, we obtain

$$\int_{D_T} (\chi - \mathcal{K}(\theta))\psi \ dxdt \geq 0, \quad \forall\psi \in L^\infty(0,T;L^4(D)) \tag{127}$$

and we conclude that $\chi = \mathcal{K}(\theta)$. □

4.3 End of proof of theorem 1

Let $(U_\nu, M_\nu, H_\nu, \theta_\nu, Q_\nu)$ be the solution of problem $(\mathcal{P}_\nu)$ provided by Theorem 2 for $\nu > 0$. The energy estimates (46) and (48) lead to the following uniform bounds with respect to ν

Lemma 14

- *(U_ν) is uniformly bounded in $L^\infty(0,T;\mathcal{U}_0) \cap L^2(0,T;\mathcal{U})$*

- *(M_ν) and (H_ν) are uniformly bounded in $L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1_t(D))$*
- *$(M_\nu \times H_\nu)$ is uniformly bounded in $L^2(0,T;\mathbb{L}^2(D))$*
- *(Q_ν) is uniformly bounded in $L^\infty(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1_n(D))$*

To deal with the nonlinear terms of the equations, we need to estimate the time derivatives $(\partial_t U_\nu), (\partial_t M_\nu)$ and $(\partial_t Q_\nu)$. For the first one, we deduce from the weak formulation of the equation of U_ν that in $\mathcal{D}'(D)$, p.p. $t \in (0,T)$ it holds

$$\partial_t U_\nu = -\mathcal{L}_\nu \tag{128}$$

where the linear form $\mathcal{L}_\nu$ is defined on $\mathcal{U}$ by

$$\begin{aligned}\langle \mathcal{L}_\nu, \varphi\rangle = &\int_D (U_\nu \cdot \nabla) U_\nu \cdot \varphi \, dx + \eta \int_D \nabla U_\nu \cdot \nabla \varphi \, dx + \int_D \rho(\theta_\nu) g \cdot \varphi \, dx \\ &-\mu_0 \int_D (M_\nu \cdot \nabla) H_\nu \cdot \varphi \, dx - \frac{\mu_0}{2} \int_D M_\nu \times H_\nu \cdot \operatorname{curl} \varphi \, dx.\end{aligned} \tag{129}$$

Therefore $\mathcal{L}_\nu \in \mathcal{U}'$ p.p. $t \in (0,T)$ and

$$\|\mathcal{L}_\nu\|_{\mathcal{U}'} \leq C(\|U_\nu\|^2_{H^1(D)} + \|U_\nu\|_{H^1(D)} + \|\rho(\theta_\nu)\| +$$

$$\|M_\nu\|_{H^1(D)} \|H_\nu\|_{H^1(D)} + \|M_\nu \times H_\nu\|) \leq C$$

thanks to the bounds of Lemma 14. We deduce that $\partial_t U_\nu$ is uniformly bounded in $L^1(0,T;\mathcal{U}')$. and since the same proofs work for $\partial_t M_\nu$ and $\partial_t Q_\nu$, we get

Lemma 15 *$(\partial_t U_\nu), (\partial_t M_\nu)$ and $(\partial_t Q_\nu)$ are uniformly bounded with respect to ν in $L^1(0,T;\mathcal{U}')$, $L^1(0,T;(H^1_t(D))')$ and $L^1(0,T;(H^1_n(D))')$ respectively.*

Hence we get the convergence results

Lemma 16 *There exists subsequences still denoted (U_ν), (M_ν), (H_ν), (Q_ν) such that when $\nu \to 0$*

$$\begin{aligned}&U_\nu \rightharpoonup U \text{ weakly} - \star \text{ in } L^\infty(0,T;\mathcal{U}_0) \text{ and weakly in } L^2(0,T;\mathcal{U})\\ &M_\nu \rightharpoonup M \text{ weakly} - \star \text{ in } L^\infty(0,T;\mathbb{L}^2(D)) \text{ and weakly in } L^2(0,T;\mathbb{H}^1_t(D))\\ &H_\nu \rightharpoonup H \text{ weakly} - \star \text{ in } L^\infty(0,T;\mathbb{L}^2(D)) \text{ and weakly in } L^2(0,T;\mathbb{H}^1_t(D))\\ &Q_\nu \rightharpoonup Q \text{ weakly} - \star \text{ in } L^\infty(0,T;\mathbb{L}^2(D)) \text{ and weakly in } L^2(0,T;\mathbb{H}^1_n(D))\end{aligned} \tag{130}$$

and the following strong convergence results hold

$$(U_\nu,\ M_\nu,\ H_\nu,\ Q_\nu) \to (U,\ M,\ H,\ Q) \text{ strongly in } (\mathbb{L}^2(D_T))^4 \tag{131}$$

where $H = \mathcal{H}(M,F)$.

Once again, the strong convergence of (H_ν) is a consequence of subsection 1.3.

Notice that from Lemma 16, the hypotheses (120) are fulfilled leading to the convergence of the temperature of the previous subsection. Proceeding as for the proof of Theorem

2, we can verify that (U, M, H, θ, Q) is a weak solution with finite energy of problem ($\mathcal{P}$) which leads to the conclusion of the first part of Theorem 1.

Let us prove the second part of the Theorem , which is the $L^p - L^q$- regularity for the temperature θ. In this purpose, it is enough to show that

Lemma 17 *Under hypotheses (36), the regularized temperature θ_ν is uniformly bounded in $L^{36/11}(0,T;L^{36/7}(D))$ with respect to ν.*

Proof. From (117) and using the Sobolev embedding $W^{1,12/11}(D) \subset L^{12/7}(D)$, we see that (ζ_ν) is uniformly bounded in $L^{12/11}(0,T;L^{12/7}(D))$ then from the relation $\alpha\theta_\nu^3 = -\zeta_\nu + \tau\gamma \mathrm{div}\, Q_\nu - \kappa\theta_\nu$, we deduce that (θ_ν) is uniformly bounded in $L^{36/11}(0,T;L^{36/7}(D))$. □

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